

Discrete signals

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Discrete signals

Inner products and energy

Discrete complex exponentials

- ▶ Discrete and finite time index $n = 0, 1, \dots, N - 1 = [0, N - 1]$.

- ▶ Discrete signal x is a **function mapping** $[0, N - 1]$ to a real **value** $x(n)$

$$x : [0, N - 1] \rightarrow \mathbb{R}$$

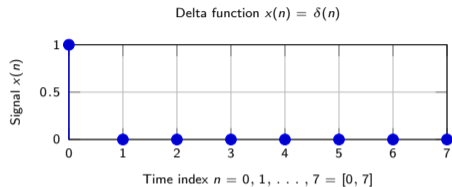
- ▶ The values that the signal takes at time index n is $x(n)$
- ▶ Sometimes it will make sense to talk about complex signals

$$x : [0, N - 1] \rightarrow \mathbb{C}$$

- ▶ The values $x(t) = x_R(t) + j x_I(t)$ are complex numbers
- ▶ Space of signals = space of N -dimensional vectors \mathbb{R}^N or \mathbb{C}^N

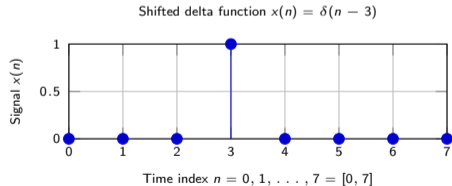
- ▶ The discrete delta function $\delta(n)$ is a spike at (initial) time $n = 0$

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else} \end{cases}$$



- ▶ The shifted delta function $\delta(n - n_0)$ has a spike at time $n = n_0$

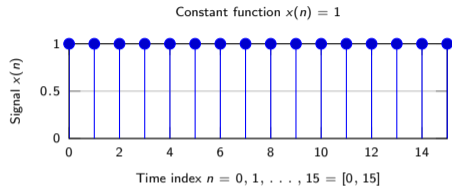
$$\delta(n - n_0) = \begin{cases} 1 & \text{if } n = n_0 \\ 0 & \text{else} \end{cases}$$



- ▶ This is not a new definition, just a time shift

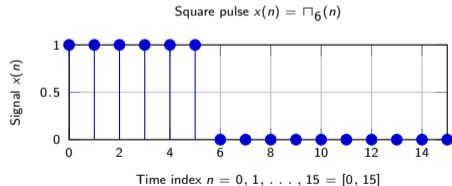
- ▶ A constant function $x(n)$ has the same value c for all n

$$x(n) = c, \quad \text{for all } n$$



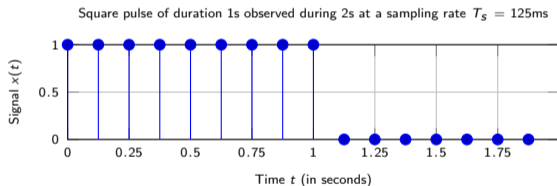
- ▶ A square pulse of width M , $\Pi_M(n)$, equals one for the first M values

$$\Pi_M(n) = \begin{cases} 1 & \text{if } 0 \leq n < M \\ 0 & \text{if } M \leq n \end{cases}$$



- ▶ Can consider shifted pulses $\Pi_M(n - n_0)$, with $n_0 < N - M$

- ▶ **Sampling time T_s** \Rightarrow Time elapsed between indexes n and $n + 1$
 - \Rightarrow Sampling frequency $f_s := 1/T_s$
- ▶ Time index n represents actual time $t = nT_s$

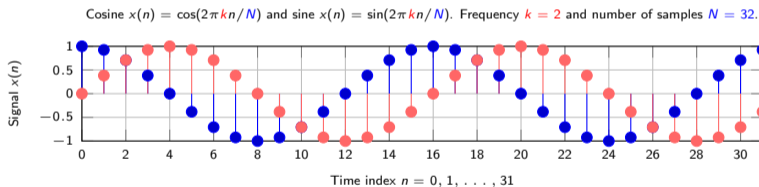


- ▶ Signal duration $T = NT_s$ \Rightarrow Time length of signal
 - \Rightarrow The last sample is “held” during T_s time units

- ▶ For a signal of duration N define (assume N is even):

⇒ Discrete cosine of discrete frequency $k \Rightarrow x(n) = \cos(2\pi kn/N)$

⇒ Discrete sine of discrete frequency $k \Rightarrow x(n) = \sin(2\pi kn/N)$

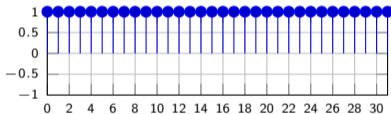


- ▶ Frequency k is discrete. I.e., $k = 0, 1, 2, \dots$

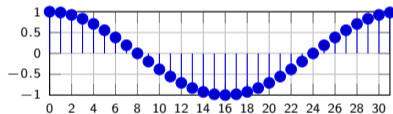
⇒ Have an integer number of complete oscillations

- ▶ Discrete frequency $k = 0$ is a constant
- ▶ Discrete frequency $k = 1$ is a complete oscillation
- ▶ Frequency $k = 2$ is two oscillations, for $k = 3$ three oscillations ...

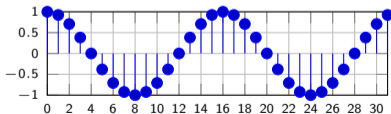
Frequency $k = 0$. Number of samples $N = 32$



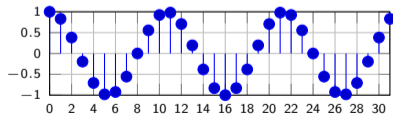
Frequency $k = 1$. Number of samples $N = 32$



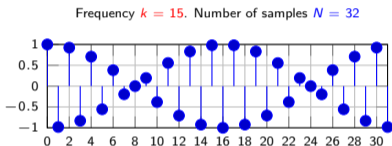
Frequency $k = 2$. Number of samples $N = 32$



Frequency $k = 3$. Number of samples $N = 32$



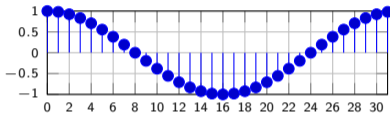
- ▶ Frequency k represents k complete oscillations
- ▶ Although for large k the oscillations may be difficult to see



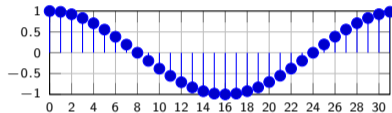
- ▶ Do note that we can't have more than $N/2$ oscillations
 - ⇒ Indeed $1 \rightarrow -1 \rightarrow 1, \rightarrow -1, \dots$
 - ⇒ Frequency $N/2$ is the last one with physical meaning
- ▶ Larger frequencies replicate frequencies between $k = 0$ and $k = N/2$

- Frequencies k and $N - k$ represent the same cosine

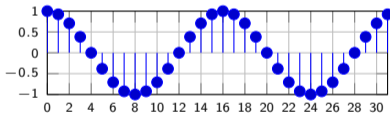
Frequency $k = 1$. Number of samples $N = 32$



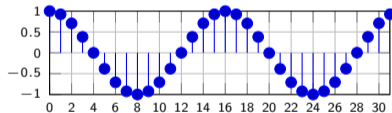
Frequency $N - k = 31$. Number of samples $N = 32$



Frequency $k = 2$. Number of samples $N = 32$



Frequency $N - k = 30$. Number of samples $N = 32$



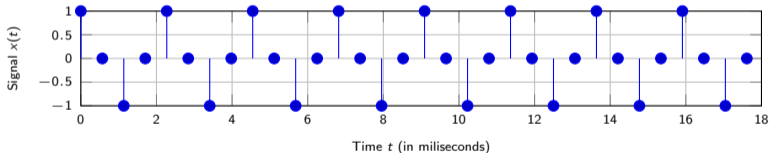
- Actually, if $k + l = \dot{N}$, cosines of frequencies k and l are equivalent
- Not true for sines, but almost. The signals have opposite signs

- ▶ What is the **discrete frequency** k of a cosine of **frequency** f_0 ?
- ▶ Depends on sampling time T_s , frequency $f_s = \frac{1}{T_s}$, duration $T = NT_s$
- ▶ Period of discrete cosine of frequency k is T/k (k oscillations)
- ▶ Thus, regular frequency of said cosine is $\Rightarrow f_0 = \frac{k}{T} = \frac{k}{NT_s} = \frac{k}{N}f_s$
- ▶ A cosine of **frequency** f_0 has discrete **frequency** $k = (f_0/f_s)N$
- ▶ Only frequencies up to $N/2 \leftrightarrow f_s/2$ have physical meaning
- ▶ **Sampling frequency** $f_s \Rightarrow$ Cosines up to frequency $f_0 = f_s/2$

- ▶ Generate $N = 32$ samples of an A note with sampling frequency $f_s = 1,760\text{Hz}$
- ▶ The frequency of an A note is $f_0 = 440\text{Hz}$. This entails a discrete frequency

$$k = \frac{f_0}{f_s} N = \frac{440\text{Hz}}{1,760\text{Hz}} 32 = 8$$

The A note observed during $T = NT_s = 18.2\text{ms}$ with a sampling rate $f_s = 1,760\text{Hz}$



- ▶ Alternatively $\Rightarrow x(n) = \cos\left[2\pi kn/N\right] = \cos\left[2\pi(f_0/f_s)Nn/N\right]$
- ▶ Which simplifies to $\Rightarrow x(n) = \cos\left[2\pi(f_0/f_s)n\right] = \cos\left[2\pi f_0(nT_s)\right]$

- ▶ The frequency k need not be integer but it's not a discrete cosine

⇒ Sampled cosine ⇒ $x(n) = \cos(2\pi kn/N)$

⇒ Sampled sine ⇒ $x(n) = \sin(2\pi kn/N)$

- ▶ Discrete sine and cosine have complete oscillations
- ▶ Sampled sine and cosine may have incomplete oscillations
- ▶ **Discrete** sine and cosine are used to define **Fourier transforms** (later)

Discrete signals

Inner products and energy

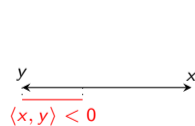
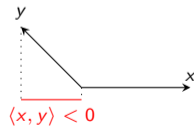
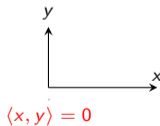
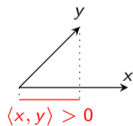
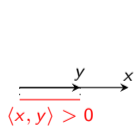
Discrete complex exponentials

- ▶ Given two signals x and y with components $x(n)$ and $y(n)$ define the **inner product** of x and y as

$$\begin{aligned}
 \langle x, y \rangle &:= \sum_{n=0}^{N-1} x(n)y^*(n) \\
 &= \sum_{n=0}^{N-1} x_R(n)y_R(n) - \sum_{n=0}^{N-1} x_I(n)y_I(n) + j \sum_{n=0}^{N-1} x_I(n)y_R(n) + j \sum_{n=0}^{N-1} x_R(n)y_I(n)
 \end{aligned}$$

- ▶ This is the same as the inner product between vectors x and y . Just with different notation
- ▶ The Inner product is a linear operations $\Rightarrow \langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
- ▶ Reversing the order of the factor results in conjugation $\Rightarrow \langle y, x \rangle = \langle x, y \rangle^*$

- ▶ The inner product $\langle x, y \rangle$ is the projection of the signal (vector) y on the signal (vector) x
- ▶ The value of $\langle x, y \rangle$ is how much of y falls in x direction
 - ⇒ How much y resembles x . How much x predicts y . Knowing x , how much of y we know
 - ⇒ Very importantly, if $\langle x, y \rangle = 0$ the signals are **orthogonal**. They are “unrelated”



► Define the **norm** of signal x as $\Rightarrow \|x\| := \left[\sum_{n=0}^{N-1} |x(n)|^2 \right]^{1/2} = \left[\sum_{n=0}^{N-1} |x_R(n)|^2 + \sum_{n=0}^{N-1} |x_I(n)|^2 \right]^{1/2}$

► Define the **energy** as the norm squared $\Rightarrow \|x\|^2 := \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} |x_R(n)|^2 + \sum_{n=0}^{N-1} |x_I(n)|^2$

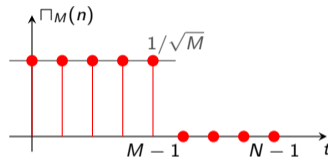
► The energy of x is the inner product of x with itself $\Rightarrow \|x\|^2 = \langle x, x \rangle$

► Recall that for complex numbers we have $x(n)x^*(n) = |x_R(n)|^2 + |x_I(n)|^2 = |x(n)|^2$

- ▶ Inner product can't exceed the product of the norms $\Rightarrow -\|x\| \|y\| \leq \langle x, y \rangle \leq \|x\| \|y\|$
- ▶ Inner product squared can't exceed product of energies $\Rightarrow \langle x, y \rangle^2 \leq \|x\|^2 \|y\|^2$
- ▶ If you prefer explicit expressions $\Rightarrow \sum_{n=0}^{N-1} x(n)y^*(n) \leq \left[\sum_{n=0}^{N-1} |x(n)|^2 \right] \left[\sum_{n=0}^{N-1} |y(n)|^2 \right]$
- ▶ The equalities hold if and only if the signals (vectors) x and y are collinear (aligned)

- ▶ The unit energy square pulse is the signal $\Pi_M(n)$ that takes values

$$\begin{aligned} \Pi_M(n) &= \frac{1}{\sqrt{M}} && \text{if } 0 \leq n < M \\ \Pi_M(n) &= 0 && \text{if } M \leq n \end{aligned}$$

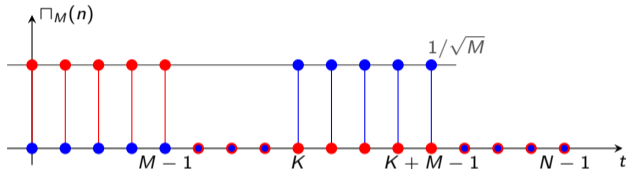


- ▶ To compute energy of the pulse we just evaluate the definition

$$\|\Pi_M\|^2 := \sum_{n=0}^{N-1} |\Pi_M(n)|^2 = \sum_{n=0}^{M-1} \left| \frac{1}{\sqrt{M}} \right|^2 = \frac{M}{M} = 1$$

- ▶ As name indicates, the unit energy square pulse has unit energy . If pulse height is 1, energy is M .

- ▶ Shift pulse by modifying argument $\Rightarrow \Pi_M(n - K) \Rightarrow$ Pulse is now centered at K



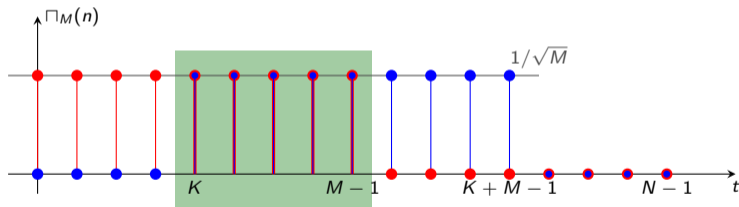
- ▶ If the pulse support is disjoint ($K \geq M$), the inner product of two pulses is zero

$$\langle \Pi_M(n), \Pi_M(n - K) \rangle := \sum_{n=0}^{N-1} \Pi_M(n) \Pi_M(n - K) = 0$$

- ▶ Pulses are orthogonal \Rightarrow They are “unrelated.” One pulse does not predict the other

- ▶ If $K < M$ the pulses overlap. They overlap between $n = K$ and $n = M - 1$. Thus, the inner product is

$$\langle \Pi_M(n), \Pi_M(n - K) \rangle := \sum_{n=0}^{N-1} \Pi_M(n) \Pi_M(n - K) = \sum_{n=K}^{M-1} \left(\frac{1}{\sqrt{M}} \right) \left(\frac{1}{\sqrt{M}} \right) = \frac{M - K}{M} = 1 - \frac{K}{M}$$



- ▶ Inner product proportional to relative overlap \Rightarrow How much the pulses are “related” to each other

Discrete signals

Inner products and energy

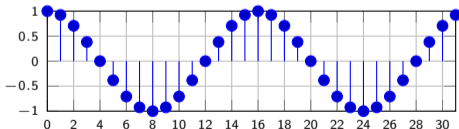
Discrete complex exponentials

- ▶ Discrete complex exponential of discrete frequency k and duration N

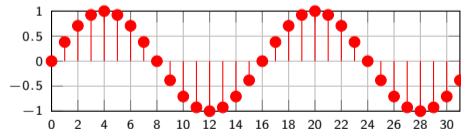
$$e_{kN}(n) = \frac{1}{\sqrt{N}} e^{j2\pi kn/N} = \frac{1}{\sqrt{N}} \exp(j2\pi kn/N)$$

- ▶ The complex exponential function is $\Rightarrow e^{j2\pi kn/N} = \cos(2\pi kn/N) + j \sin(2\pi kn/N)$
- ▶ The Real part is a discrete cosine. The imaginary part a discrete sine. An oscillation

$\text{Re}(e^{j2\pi kn/N})$, with $k = 2$ and $N = 32$



$\text{Im}(e^{j2\pi kn/N})$, with $k = 2$ and $N = 32$



[P1] For frequency $k = 0$, the exponential $e_{kN}(n) = e_{0N}(n)$ is a constant $\Rightarrow e_{kN}(n) = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}} \mathbf{1}$

[P2] For frequency $k = N$, the exponential $e_{kN}(n) = e_{NN}(n)$ is a constant. True for any multiple $k \in \dot{N}$

$$e_{NN}(n) = \frac{e^{j2\pi Nn/N}}{\sqrt{N}} = \frac{(e^{j2\pi})^n}{\sqrt{N}} = \frac{(1)^n}{\sqrt{N}} = \frac{1}{\sqrt{N}}$$

[P3] For $k = \frac{N}{2}$, the exponential $e_{kN}(n) = e_{N/2N}(n) = (-1)^n / \sqrt{N}$. Fastest possible oscillation with N samples

$$e_{N/2N}(n) = \frac{e^{j2\pi(N/2)n/N}}{\sqrt{N}} = \frac{(e^{j\pi})^n}{\sqrt{N}} = \frac{(-1)^n}{\sqrt{N}}$$

That $e^{j2\pi} = 1$ follows from $e^{j\pi} = -1$. Which follows from $e^{j\pi} + 1 = 0$. Relates five most important constants in mathematics.

Theorem

If the *frequency difference is* $k - l = N$ the signals $e_{kN}(n)$ and $e_{lN}(n)$ coincide for all n , i.e.,

$$e_{kN}(n) = \frac{e^{j2\pi kn/N}}{\sqrt{N}} = \frac{e^{j2\pi ln/N}}{\sqrt{N}} = e_{lN}(n)$$

- ▶ **Exponentials** with frequencies k and l are **equivalent** if the **frequency difference is** $k - l = N$

Proof.

- ▶ We prove by showing that the ratio $e_{kN}(n)/e_{lN}(n) = 1$. Combine exponents

$$\frac{e_{kN}(n)}{e_{lN}(n)} = \frac{e^{j2\pi kn/N}}{e^{j2\pi ln/N}} = e^{j2\pi(k-l)n/N}$$

- ▶ By hypothesis we have that $k - l = N$. Therefore, the latter simplifies to

$$\frac{e_{kN}(n)}{e_{lN}(n)} = e^{j2\pi Nn/N} = \left[e^{j2\pi} \right]^n = 1^n = 1$$

□

- ▶ Canonical set \Rightarrow Suffice to look at N consecutive frequencies, e.g., $k = 0, 1, \dots, N - 1$

$$\begin{array}{cccc}
 -N, & -N + 1, & \dots, & -1 \\
 0, & 1, & \dots, & N - 1 \\
 N, & N + 1, & \dots, & 2N - 1
 \end{array}$$

- ▶ Another canonical choice is to make $k = 0$ a center frequency

$$\begin{array}{ccccccc}
 -N/2 + 1, & \dots, & -1, & 0, & \dots, & N/2 \\
 N/2 + 1, & \dots, & N - 1, & N, & \dots, & 3N/2
 \end{array}$$

- ▶ With N even (as usual) we use $N/2$ positive frequencies and $N/2 - 1$ negative frequencies
- ▶ From one canonical set to the other \Rightarrow Chop and shift

Theorem

Complex exponentials with nonequivalent frequencies are orthogonal. I.e.

$$\langle \mathbf{e}_{kN}, \mathbf{e}_{lN} \rangle = 0$$

when $k - l < N$. E.g., when $k = 0, \dots, N - 1$, or $k = -N/2 + 1, \dots, N/2$.

- ▶ Signals of canonical sets are “unrelated.” Different rates of change
- ▶ Also note that the energy is $\|\mathbf{e}_{kN}\|^2 = \langle \mathbf{e}_{kN}, \mathbf{e}_{kN} \rangle = 1$
- ▶ Exponentials with frequencies $k = 0, 1, \dots, N - 1$ are orthonormal

$$\langle \mathbf{e}_{kN}, \mathbf{e}_{lN} \rangle = \delta(l - k)$$

- ▶ They are an **orthonormal basis** of signal space with N samples

Proof.

- ▶ Use definitions of inner product and discrete complex exponential to write

$$\langle e_{kN}, e_{lN} \rangle = \sum_{n=0}^{N-1} e_{kN}(n) e_{lN}^*(n) = \sum_{n=0}^{N-1} \frac{e^{j2\pi kn/N}}{\sqrt{N}} \frac{e^{-j2\pi ln/N}}{\sqrt{N}}$$

- ▶ Regroup terms to write as geometric series

$$\langle e_{kN}, e_{lN} \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N} = \frac{1}{N} \sum_{n=0}^{N-1} \left[e^{j2\pi(k-l)/N} \right]^n$$

- ▶ Geometric series with basis a sums to $\sum_{n=0}^{N-1} a^n = (1 - a^N)/(1 - a)$. Thus,

$$\langle e_{kN}, e_{lN} \rangle = \frac{1}{N} \frac{1 - \left[e^{j2\pi(k-l)/N} \right]^N}{1 - e^{j2\pi(k-l)/N}} = \frac{1}{N} \frac{1 - 1}{1 - e^{j2\pi(k-l)/N}} = 0$$

- ▶ Completed proof by noting $\left[e^{j2\pi(k-l)/N} \right]^N = e^{j2\pi(k-l)} = \left[e^{j2\pi} \right]^{(k-l)} = 1$ □

Theorem

Opposite frequencies k and $-k$ yield conjugate signals: $e_{-kN} = e_{kN}^*$

Proof.

- ▶ Just use the definitions to write the chain of equalities

$$e_{-kN}(n) = \frac{e^{j2\pi(-k)n/N}}{\sqrt{N}} = \frac{e^{-j2\pi kn/N}}{\sqrt{N}} = \left[\frac{e^{j2\pi kn/N}}{\sqrt{N}} \right]^* = e_{kN}^*(n) \quad \square$$

- ▶ Opposite frequencies \Rightarrow Same real part. Opposite imaginary part

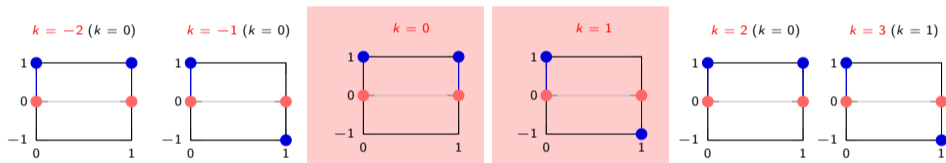
\Rightarrow The cosine is the same, the sine changes sign

- ▶ Of N canonical frequencies, only $\frac{N}{2} + 1$ are distinct. No more than $\frac{N}{2}$ oscillations in N samples

$$\begin{array}{cccc}
 0, & 1, & \dots, & N/2 - 1 \\
 & -1, & \dots, & -N/2 + 1 \\
 & N - 1, & \dots, & N/2 + 1
 \end{array} \quad N/2$$

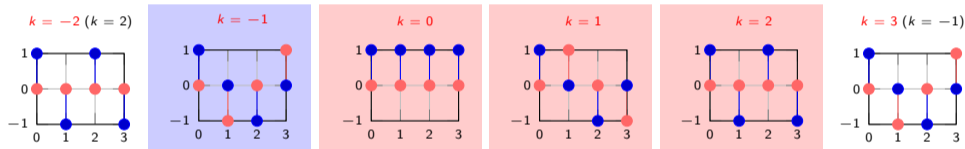
- ▶ The frequencies 0 and $N/2$ do not have a conjugate counterpart. All Others do
- ▶ The canonical set $-N/2 + 1, \dots, -1, 0, 1, \dots, N/2$ is easier to interpret
 - ⇒ Positive frequencies ranging from 0 to $N/2 \leftrightarrow f_s/2$ have physical meaning
 - ⇒ The negative frequencies are conjugates of the corresponding positive frequencies

- ▶ When signal duration is $N = 2$ only frequencies $k = 0$ and $k = 1$ represent distinct signals



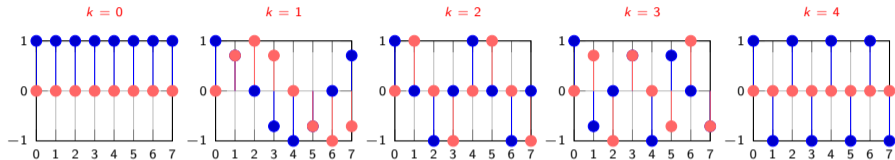
- ▶ The signals are real, they have no imaginary parts

- ▶ When $N = 4$, $k = 0, 1, 2$ are distinct. $k = -1$ is conjugate of $k = 1$

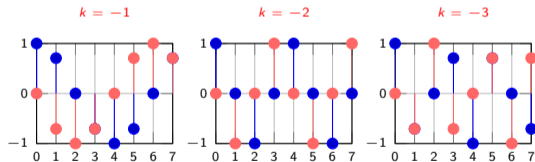


- ▶ Can also use $k = 3$ as canonical instead of $k = -1$ (conjugate of $k = 1$)

- Frequencies from $k = 1$ to $k = 4$ represent distinct signals



- Frequencies $k = -1$ to $k = -3$ are conjugate signals of $k = 1$ to $k = 3$



- All other frequencies represent one of the signals above

- There are 9 distinct frequencies and 7 conjugates (not shown). Some look like actual oscillations

